Credit Risk Models IV: Understanding and pricing CDOs

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December 2005†

Abstract

Some investors in the Collateralized Debt Obligations (CDOs) market have been publicly accused of not fully understanding the risks and dynamics of these products. They won't have an excuse any more. This report explains the mechanics of CDOs: their implied cash flows, the variables affecting those cash flows, their pricing, the sensitivity of CDO prices to those variables, the functioning of the markets where they are traded, their different types, the conventions used for trading CDOs, ... We built our description of CDOs pricing upon the Vasicek asymptotic single factor model because of its simplicity and the insights it provides regarding the pricing of CDOs. Additionally, we provide an extensive and updated review of the literature which extends the Vasicek model by relaxing its, somehow restrictive, assumptions in order to build more realistic and, as a consequence, more complicated CDO pricing models.

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†First version: July 2005. An updated version is available at `www.abelelizalde.com`.

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This paper is part of a series of surveys on credit risk modelling and pricing. The complete list of surveys is available at `www.abelelizalde.com`, and consists on the following:

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5. Credit Default Swap Valuation: An Application to Spanish Firms.

“**there is a minority of investors - perhaps 10 per cent - who do not fully understand what they are getting into**.”

Michael Gibson, head of trading risk analysis at the US Federal Reserve (Financial Times, 2005a).

“**Understanding the credit risk profile of CDO tranches poses challenges even to the most sophisticated participants.**”

Alan Greenspan, chairman of the US Federal Reserve (Financial Times, 2005b).

The “sharp increase in the complexity of credit derivative products being traded in the past couple of years ... may also mean that **investors do not fully understand what they are purchasing in areas such as collateralised debt obligations (CDOs)** - or pools of debt linked securities.”

(Financial Times, 2005c).

“Last month, Bank of America and Italian bank Banca Popolare di Intra (BPI) settled their 40 million euro lawsuit, in which BPI claims it was mis-sold several CDO investments by Bank of America. ... It would be naive to think that this is the last court case that will emerge. **A number of investors and regulators have already voiced concern about the level of complexity in some investment products.** With something as complicated as CDO-squared, it’s not hard to imagine more investors claiming they were mis-sold investments if the credit cycle takes a turn for the worse.”

Nick Sawyer, Editor (Risk, 2005).

1 Introduction

Imagine a pool of defaultable instruments (bonds, loans, credit default swaps CDSs, ...) from different firms is put together. The losses on the initial portfolio value due to the default of the underlying firms depend on the default probability of each firm and the losses derived from each default (losses given default). Additionally, the degree of dependence between the firms' default probability, usually known as default correlation, plays an important role on the timing of the firms' defaults (whether they tend to cluster or they are independent) and, as a consequence, on the distribution of the portfolio losses.

Next, imagine we, the owners of the portfolio, decide to buy protection against the possible losses due to the defaults of the underlying firms, but we can not sell the portfolio. One way to do it is buying Credit Default Swaps (CDSs) of each firm, but that's not the way we are interested in here. We can *sell the portfolio in tranches*, i.e. we can buy protection for those losses in tranches. A Collateralized Debt Obligation (CDO) consists on *tranching* and selling the credit risk of the underlying portfolio. For example, a tranche with attachment points $[K_L, K_U]$ will bear the portfolio losses in excess of K_L percent of the initial value of the portfolio, up to a K_L percent. The tranche absorbing the first losses, called equity tranche, is characterized by $K_L = 0$ and $K_U > 0$. The holders of a tranche characterized by attachment points $[K_L, K_U]$ won't suffer any loss as long as the total portfolio loss is lower than K_L percent of its initial value. When the total portfolio loss goes above K_L percent, the tranche holders are responsible for the losses exceeding K_L percent, up to K_U percent. Losses above K_U percent of the initial portfolio value do not affect them. The lower attachment point K_L of each tranche corresponds to the upper attachment point K_U of the previous (more junior) tranche.

Obviously, the holders of each tranche (sellers of credit risk protection) have to be compensated for bearing those losses: they receive a periodic fee, called premium, until the maturity of the CDO (point in which they also stop being responsible for future losses in the portfolio.) The premium of the equity tranche will be the highest because its holders absorb the first losses of the portfolio. In order for the holders of more senior tranches to start suffering losses, the holders of more junior tranches would have already born all losses they were exposed to ($K_U - K_L$ percent of the initial portfolio value). As a consequence, the higher the seniority of the tranche the lower the premiums holders receive.

The whole problem lies in determining the tranches' premiums. They have to compensate tranche holders for the expected losses they will suffer and, therefore, they depend on the distribution of the portfolio losses which, as we argued above, depends on the underlying firms' default probabilities, default correlations, and losses given default.

Our review of CDO pricing models focus on a particular branch of this literature: the ones based on structural models. The main distinguishing characteristic of such models with respect to the other credit risk modelling alternative, reduced form models, is the link they provide between the probability of default and the firms' fundamental financial variables: assets and liabilities. The way structural models incorporate the dependence between the firms' default probabilities (which is a key ingredient for CDO pricing) is by making such fundamental variables depend on a set of, generally unobserved, common factors.

In contrast, reduced form models rely on market prices of the firms' defaultable instruments to extract both their default probabilities and their credit risk dependencies. These models rely on the market as the only source of information regarding the firms' credit structure and do not consider any information coming from their balance sheets. Although easier to calibrate, reduced form models lack the link between credit risk and the firms' financial situation incorporated in their assets and liabilities. Anyway, reduced form models provide an alternative way of pricing CDOs which one shouldn't forget, besides their lower popularity in this area.[1] In fact, these

[1]See, among others, Chava and Jarrow (2004), Driessen (2005), and Elizalde (2005d) for intensity models incorporating the correlation structure across firms, and Galiani (2003), Duffie and Garleanu

models provide the dynamics needed to price some of the recent exotic CDO products which we review in Section 4.5.4.

The paper starts from the theoretical foundations of the Vasicek model. It presents, in Section 2, a review of credit risk structural models, in order to understand the motivations behind such models. Section 3 describes in detail the Vasicek asymptotic single risk factor model, which has become the market standard for CDO pricing, and which is also referred to as the normal or Gaussian copula model, because that is the dependence structure it implies for the firms' default correlation.[2]

With all those tools in hand, Section 4 dives into CDOs: mechanics, types, pricing, premium sensitivity to the model parameters, trading issues (implied and base correlations), extensions of the Vasicek model and, finally, a few words on the calibration of the reviewed models.

The assumptions of the Vasicek asymptotic single risk factor model about the characteristics of the underlying portfolio (homogeneous infinitely large portfolio, ...) simplify the analytical derivation of CDO premiums but are not very realistic. The extensions we present relax these assumptions, making the model more suitable for CDO pricing. Investment banks and rating agencies devote a large amount of effort (and money) to fine tune and improve such models in order to benefit from the increase in pricing accuracy.

The text ends with an Appendix describing the application of the Vasicek factor model to bank capital regulation in Basel II, the brand new accord for banking supervision and regulation. Although this is not directly related with the main topic of the text, we include it because (i) it might interest some readers, (ii) it is the other most popular application of the Vasicek model, and (iii) it is straightforward using the material presented in Sections 2 and 3.

Throughout the text we provide an extensive list of references which the reader

(2004), and Willemann (2004) for applications of CDO pricing using intensity based methods via Monte Carlo simulation of default times.

[2] See Nelsen (1999), Embrechts, McNeil and Straumann (2002), and Cherubini and Luciano (2004) for an analysis of copula functions.

further interested in any of the covered topics might find useful.

2 Structural model for credit risk: Merton (1974)

There are two primary types of models that attempt to describe default processes in the credit risk literature: structural and reduced form models.[3]

Structural models use the evolution of firms' structural variables, such as asset and debt values, to determine the time of default. Merton's model (1974) was the first modern model of default and is considered the first structural model. In Merton's model a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm's asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

Reduced form models do not consider the relationship between default and firm financial situation in an explicit manner. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data.

Structural default models provide a link between the credit quality of a firm and the firm's economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced approach.

Merton (1974) makes use of the Black and Scholes (1973) option pricing model to value corporate liabilities. As we shall see, this is an straightforward application only if we adapt the firm's capital structure and the default assumptions to the requirements of the Black-Scholes model.

Assume that the dynamics of firm n's asset value $A_{n,t}$ follow a continuous-time diffusion given, under the physical or real probability measure $\mathbf{P}$, by the following

[3] For a literature review of credit risk models see Elizalde (2005b and c).

geometric Brownian motion:

$$\frac{dA_{n,t}}{A_{n,t}} = \mu_n dt + \sigma_n dW_{n,t}, \tag{1}$$

where μ_n is the total expected return, σ_n is the asset's (relative) instantaneous volatility, and $W_{n,t}$ is a standard Brownian motion under $\mathbf{P}$.

Let us assume that the capital structure of firm n is comprised by equity and by a zero-coupon bond with maturity T and face value of D_n. The firm's asset value $A_{n,t}$ is simply the sum of equity and debt values. Under these assumptions, equity represents a call option on the firm's assets with maturity T and strike price of D_n. It is assumed that the firm defaults if, at maturity T the firm's asset value $A_{n,T}$ is not enough to pay back the face value of the debt D_n to bondholders. As a consequence, the probability at time $t < T$ of the firm defaulting at T is given by

$$p_{n,t,T} = \mathbf{P}\left[A_{n,T} < D_n \mid A_{n,t}\right]. \tag{2}$$

This approach assumes that default can only happen at the maturity of the zero-coupon bond.

It can be shown using Itô's lemma that the diffusion process (1) allows us to express the asset value at time T as a function of the current asset value $A_{n,t}$ as follows

$$A_{n,T} = A_{n,t} \exp\left\{\left(\mu_n - \frac{\sigma_n^2}{2}\right)(T-t) + \sigma_n\sqrt{T-t}X_{n,t,T}\right\}, \tag{3}$$

where $X_{n,t,T}$ is given by

$$X_{n,t,T} = \frac{W_{n,T} - W_{n,t}}{\sqrt{T-t}}, \tag{4}$$

and follows a standard normal distribution with zero mean and variance one.[4]

At time t, we can express the condition for firm n defaulting at time T in terms of the random variable $X_{n,t,T}$:

$$A_{n,T} < D_n \Leftrightarrow X_{n,t,T} < K_{n,t,T}, \tag{5}$$

[4] By definition of a Brownian motion, the difference $W_{n,T} - W_{n,t}$ follows a normal distribution with zero mean and standard deviation $\sqrt{T-t}$.

where

$$K_{n,t,T} = \frac{\ln D_n - \ln A_{n,t} - \left(\mu_n - \frac{\sigma_n^2}{2}\right)(T-t)}{\sigma_n \sqrt{T-t}}. \tag{6}$$

As a consequence we can rewrite (2) as

$$p_{n,t,T} = \Phi\left(K_{n,t,T}\right), \tag{7}$$

where $\Phi(\cdot)$ is the distribution function of a standard normal random variable.

Equivalently, if instead of considering the dynamics of the asset value $A_{n,t}$ under the physical probability measure $\mathbf{P}$, one considers its dynamics under the risk neutral probability measure $\mathbf{Q}$, firm n's risk neutral default probability is obtained.

In order to simplify notation hereafter we fix the actual time to $t = 0$, which allows us to eliminate the first time subindex of $p_{n,t,T}$, $X_{n,t,T}$, and $K_{n,t,T}$, which become $p_{n,T}$, $X_{n,T}$, and $K_{n,T}$.

3 Vasicek asymptotic single factor model

This Section builds on Vasicek (1987, 1991 and 2002).[5] We are interested in the default probabilities at time $t > 0$ of a group of $n = 1, ..., N$ firms with the asset and liabilities structure described in the preceding section.

The probability of default of each firm n at time t is denoted $p_{n,t}$ and given by

$$p_{n,t} = \Phi\left(K_{n,t}\right), \tag{8}$$

$$K_{n,t} = \frac{\ln D_n - \ln A_{n,0} - \left(\mu_n - \frac{\sigma_n^2}{2}\right)t}{\sigma_n \sqrt{t}}. \tag{9}$$

Imagine we have a portfolio composed of loans to each one of the above firms (one loan per firm), and we are interested in the distribution function for the *portfolio default rate*, i.e. the fraction of defaulted credits in the portfolio at time t. Note that

[5] See also Finger (1999) and Schönbucher (2000).

the portfolio default rate is not the variable we will ultimately interested in, which is the loss in the initial portfolio value or *portfolio loss rate*.

The aim (and attractiveness) of the Vasicek single factor model we are about to present is to come up with a simple and closed-form formula for the distribution function of both the portfolio default and loss rates. Deriving a closed form solution requires making a set of simplifying assumptions. We will progressively introduce these assumptions and their implications for the model.

To derive the portfolio default rate, knowing the individual probabilities $p_{1,t}, ..., p_{N,t}$ of the firms is not enough; we also need to know their correlation structure. Since the only random variable affecting the status of each firm n at time t (default or not default) is $X_{n,t}$, the correlation structure between the firms' default probabilities have to be introduced through the normal random variables $X_{1,t}, ...X_{N,t}$. We assume that the correlation coefficient of each pair of random variables $X_{n,t}$ and $X_{m,t}$ is $\rho_{n,m,t}$.

Assumption 1. The correlation coefficient $\rho_{n,m,t}$ between each pair of random variables X_n and X_m is the same for any two firms:

$$corr\left(X_{n,t}, X_{m,t}\right) = \rho_{n,m,t} = \rho_t \quad \text{for any } n \neq m. \tag{10}$$

The random part driving all firms' asset values is characterized by a common correlation coefficient. We can think that there exists a random factor or source of uncertainty affecting all firms in exactly the same way. Moreover, we can write the random variables $X_{1,t}, ..., X_{N,t}$ as

$$X_{n,t} = \sqrt{\rho_t} Y_t + \sqrt{1-\rho_t} \varepsilon_{n,t}, \tag{11}$$

for all $n = 1, ..., N$, where Y_t, $\varepsilon_{1,t}$, ..., $\varepsilon_{N,t}$ are i.i.d. standard normal random variables. We can interpret (11) as follows: each random variable $X_{n,t}$, whose realization determines whether firm n defaults at t, can be expressed as the sum of two risk factors: one common or systematic risk factor Y_t affecting all firms in the same way, and an idiosyncratic risk factor $\varepsilon_{n,t}$ independent across firms.

Conditional on the realization of the common factor Y_t the default probability at time t of each firm n is denoted by $p_n(Y_t)$ and given by

$$\begin{aligned} p_n(Y_t) &= \mathbf{P}\left[X_{n,t} < K_{n,t} \mid Y_t\right] &(12)\\ &= \mathbf{P}\left[\sqrt{\rho_t}Y_t + \sqrt{1-\rho_t}\varepsilon_{n,t} < K_{n,t} \mid Y_t\right] &(13)\\ &= \mathbf{P}\left[\varepsilon_{n,t} < \frac{K_{n,t} - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}} \mid Y_t\right] &(14)\\ &= \Phi\left(\frac{K_{n,t} - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right). &(15)\end{aligned}$$

Moreover, conditional on the value of the systematic factor Y_t, the random variables $X_{1,t}, ..., X_{N,t}$ (and the default probabilities $p_1(Y_t), ..., p_N(Y_t)$) are independent.

Assumption 2. We know the individual default probabilities of each firm defaulting at time t: $p_{1,t}, ...p_{N,t}$.

In that case, we can work out, from (8), the value of $K_{n,t}$ for each firm, $K_{n,t} = \Phi^{-1}(p_{n,t})$, and substitute it into the previous equation for the conditional default probability

$$p_n(Y_t) = \Phi\left(\frac{\Phi^{-1}(p_{n,t}) - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right). \tag{16}$$

Although the underlying theoretical model we are using is a structural one, in particular the Merton (1978) model, the previous assumption does not specify the way in which the default probabilities $p_{1,t}, ...p_{N,t}$ are computed. The underlying structural model presented in the previous section serves as an stylized theoretical foundation for the Vasicek single risk factor model. However, the default probabilities $p_{1,t}, ...p_{N,t}$ can be obtained in different ways (see Section 4.7).

Marginal default probabilities $p_{1,t}, ...p_{N,t}$ are taken as given; the Vasicek asymptotic single factor model is just a way of introducing dependence between them. Moreover, because it considers a single common factor and both common and idiosyncratic factors are normal, the Vasicek asymptotic single factor model is equivalent to a normal or Gaussian copula.[6]

[6] Alternative ways of linking the firms' default probability exist: using more factors, other dis-

Assumption 3. The default probability of all firms is the same and it is denoted by p_t:

$$p_{n,t} = p_t \quad \text{for all } n = 1, ..., N. \tag{17}$$

This assumption implies the same conditional default probability $p(Y_t)$ for all firms given the systematic risk factor Y_t

$$p(Y_t) = \Phi\left(\frac{\Phi^{-1}(p_t) - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right). \tag{18}$$

Consider, for each firm n, the random variable $L_{n,t}$ which takes value 0 if the firm has not defaulted before (or at) t and 1 otherwise. Define the random variable L_t as the sum of the random variables $L_{1,t}, ...L_{N,t}$. L_t represents the number of defaults in our portfolio.

If we divide the number of defaults in the portfolio L_t by the total number of firms N in the portfolio, we obtain the fraction of defaults in the portfolio at time t, denoted by Ω_t. Ω_t can be interpreted as the portfolio default rate at time t. The unconditional cumulative distribution function of the default rate Ω_t of a portfolio characterized by a default probability p and a correlation coefficient ρ_t is given by

$$F(\omega; p_t, \rho_t) = \mathbf{P}[\Omega_t \leq \omega]. \tag{19}$$

Assumption 4. The number of credits (one to each firm) in our portfolio is very large, $N \to \infty$.

As Schönbucher (2000) and Vasicek (2002) explain, since defaults are independent when conditioned to the realization of the common factor Y_t, the assumption of an infinitely large equal-size portfolio of credits implies that, using the law of the large numbers, the fraction of defaulted credits in the portfolio Ω_t converges to the individual default probability of each individual credit $p(Y_t)$ (assumed to be equal

tribution functions different than the normal, other copulas, ... We briefly review them in Section 4.6.2.

across credits).[7] As a consequence

$$
\begin{aligned}
F(\omega; p_t, \rho_t) &= \mathbf{P}[\Omega_t \leq \omega] && (20)\\
&= \mathbf{P}[p(Y_t) \leq \omega] && (21)\\
&= \mathbf{P}\left[\Phi\left(\frac{\Phi^{-1}(p_t) - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right) \leq \omega\right] && (22)\\
&= \mathbf{P}\left[Y_t \geq \frac{\Phi^{-1}(p_t) - \sqrt{1-\rho_t}\Phi^{-1}(\omega)}{\sqrt{\rho_t}}\right] && (23)\\
&= 1 - \Phi\left(\frac{\Phi^{-1}(p_t) - \sqrt{1-\rho_t}\Phi^{-1}(\omega)}{\sqrt{\rho_t}}\right) && (24)\\
&= \Phi\left(\frac{\sqrt{1-\rho_t}\Phi^{-1}(\omega) - \Phi^{-1}(p_t)}{\sqrt{\rho_t}}\right). && (25)
\end{aligned}
$$

When $\rho_t = 0$ defaults are statistically independent, so $\Omega_t = p_t$ with probability 1, while when $\rho_t = 1$ defaults are perfectly correlated, so $\Omega_t = 0$ with probability $1 - p_t$, and $\Omega_t = 1$ with probability p_t.

The distribution function $F(\omega; p_t, \rho_t)$ is increasing in ω, with $F(0; p_t, \rho_t) = \Phi(-\infty) = 0$ and $F(1; p_t, \rho_t) = \Phi(\infty) = 1$. Moreover, it can be shown that

$$E(\Omega_t) = p_t, \tag{26}$$

and

$$Var(\Omega_t) = \Phi_2(\Phi^{-1}(p_t), \Phi^{-1}(p_t); \rho_t) - p_t^2, \tag{27}$$

where $\Phi_2(\cdot, \cdot; \rho_t)$ is the distribution function of a zero mean bivariate normal random variable with standard deviation equal to one and correlation coefficient ρ_t; see Vasicek (2002, p.161). Therefore, the expected value of the default rate is precisely the probability of default p_t, while its variance is increasing with the correlation parameter ρ_t, with $Var(\Omega_t) = 0$ for $\rho_t = 0$ and $Var(\Omega_t) = p_t(1 - p_t)$ for $\rho_t = 1$.

Assumption 5. The loss given default on each credit, denoted by λ_t, is deterministic and the same for all firms.

[7] A more detailed proof can be found in Vasicek (1987, 1991) and Finger (1999).

λ_t is the loss on each credit due to default at t, expressed as a percentage of its size. $1 - \lambda_t$ is the so-called recovery rate.

Assumption 6. The size of each credit in the portfolio is similar.

This assumption allows a one-to-one relationship between the default rate and the loss rate or percentage loss of the total initial value of the portfolio. If a fraction Ω_t of the portfolio has defaulted by t, the percentage loss of the total initial value of the portfolio, denoted by Z_t, is $\lambda_t \Omega_t$.

4 CDOs

4.1 Mechanics

CDOs are probably the most important type of multiname credit derivative. A CDO consists on a portfolio of defaultable instruments (loans, credits, bonds or default swaps) whose credit risk is sold to investors who, in return for an agreed payment (usually a periodic fee), will bear the losses in the portfolio derived from the default of the instruments in the portfolio.

The credit risk of the portfolio underlying the CDO is sold in *tranches*. A tranche is defined by a lower and an upper *attachment points*. The buyers of the tranche with lower attachment point K_L and higher attachment point K_U will bear all losses in the portfolio value in excess of K_L, and up to K_U, percent of the initial value of the portfolio. As an example, Table 1 represents the upper and lower tranches of a fictitious CDO.

		Attachment points (%)	
Tranche number	Tranche name	Lower K_L	Upper K_U
1	Equity	0	3
2	Mezzanine 1	3	7
3	Mezzanine 2	7	10
4	Mezzanine 3	10	15
5	Senior	15	30

Table 1. Example of a CDO tranche structure.

Imagine the CDO underlying portfolio experiences a loss of 9 percent of its initial value. In that case, the holders of the equity tranche would bear the first 3% of those losses, the holders of the first mezzanine tranche would bear the next 4% of them, and the holders of the second mezzanine tranche will bear just 1% of the portfolio losses. The holders of more senior tranches (mezzanine 3 and senior) would not suffer any losses.

CDO tranching allows the holders of each tranche to limit their loss exposure to $K_U - K_L$ percent of the initial portfolio value.

Let t denote the time (in years) passed since the CDO was originated, T the maturity (in years) of the CDO, M the initial value of the portfolio, and Z_t the percentage loss in the portfolio value at time t. At time t the total loss in the portfolio value is $Z_t M$. The loss suffered by the holders of tranche j from the origination (at time 0) of the CDO up to time t is a percentage $Z_{j,t}$ of the portfolio notional value M :

$$Z_{j,t} = \min\left\{Z_t, K_{U_j}\right\} - \min\left\{Z_t, K_{L_j}\right\}, \tag{28}$$

where K_{U_j} and K_{L_j} are the upper and lower attachment points of tranche j.

The losses are paid by the tranche holders during the life of the CDO, with a predetermined frequency. Let η denote such frequency in years. Usually $\eta = 0,25$, i.e. one quarter. At each payment date, tranche holders will pay the losses on the portfolio realized since the last payment date. If t was the last time in which losses were paid by tranche holders, the payment that the holders of tranche j will have to pay at time $t+\eta$ is the new loss suffered from time t : a fraction $Z_{j,t+\eta} - Z_{j,t}$ of the CDO notional M.

So far we have outlined the way in which the portfolio losses are shared among the holders of the different tranches. However, they have to be compensated for bearing the risk of such losses. The holders of tranche j receive a periodic payment, with frequency η years, equal to a premium s_j of the outstanding notional amount of tranche number j. At time t, the outstanding notional of tranche j, denoted by $\Gamma_{j,t}$, is its initial notional $\left(K_{U_j} - K_{L_j}\right) M$ minus the total losses suffered by its holders up to time t, given by $Z_{j,t}M$:

$$\begin{aligned}
\Gamma_{j,t} &= \left(K_{U_j} - K_{L_j}\right) M - Z_{j,t}M && (29) \\
&= \left(K_{U_j} - K_{L_j} - Z_{j,t}\right) M && (30) \\
& && (31) \\
&= \begin{cases} \left(K_{U_j} - K_{L_j}\right) M, & \text{if } Z_t < K_{L_j}, \\ \left(K_{U_j} - Z_t\right) M, & \text{if } K_{L_j} \leq Z_t \leq K_{U_j}, \\ 0, & \text{if } Z_t > K_{U_j}. \end{cases} && (32)
\end{aligned}$$

If $t = 0$ is the origination time, payment dates are: η, 2η, ..., T. The structure of the cash flows for the holders of tranche j is as follows. At each payment date during the life of the CDO:

- they receive an amount

$$s_j \eta \Gamma_{j,t}, \tag{33}$$

  and

- pay an amount

$$\left(Z_{j,t} - Z_{j,t-\eta}\right) M, \tag{34}$$

for $t = \eta$, 2η, ..., T.

The premium s_j does not vary during the life of the CDO. However, the notional of tranche j, $\Gamma_{j,t}$, is a decreasing function of the total portfolio losses $Z_t M$:

$$\frac{\partial \Gamma_{j,t}}{\partial \left(Z_t M\right)} = \begin{cases} 0, & \text{if } Z_t < K_{L_j}, \\ -1, & \text{if } K_{L_j} \leq Z_t \leq K_{U_j}, \\ 0, & \text{if } Z_t > K_{U_j}. \end{cases} \tag{35}$$

The outstanding notional (30) of tranche j becomes zero as soon as the percentage loss in the portfolio Z_t becomes higher than the tranche upper attachment point K_{U_j}, $Z_t \geq K_{U_j}$, which implies $Z_{j,t} = K_{U_j} - K_{L_j}$. When that happens, $\Gamma_{j,t+a\eta} = 0$ (and

$Z_{j,t+a\eta} = Z_{j,t}$) for all $a \geq 0$ and, as a consequence, the amount they have to receive (and pay) in future payment dates $t+\eta$, $t+2\eta$, ... is always zero.

The lower the seniority of the tranche, the higher are the expected losses suffered by its holders and, therefore, the higher will be the premium they receive.

A CDO whose underlying portfolio consists of credits, loans or debt instruments from different firms is called a *cash CDO*. However, the originator of a CDO, i.e. the one who buys protection (and pays the premium), does not need to physically own a portfolio of credits, loans or bonds. A portfolio of CDSs generates the same credit exposure than the portfolio of credits, loans or bonds. When the CDO is constructed using a portfolio of CDSs it receives the name of *synthetic CDO*.

Bluhm (2003) analyzes the different factors which have contributed to the success of CDO trading: spread arbitrage opportunities, regulatory capital relief, funding and economic risk transfer.

Gibson (2004) presents one of the most insightful works about the mechanics of CDOs. The author summarizes the development of CDO markets and presents a simple CDO pricing model which allows him to analyze, among other things, the risk and leverage inherent in each tranche as well as the sensitivity of each tranche to the business cycle.

Plantin (2003) presents a theoretical analysis of the rationale for tranching and securitization activities such as CDOs. The author shows that they arise as a natural profit maximizing strategy of investment banks, and predicts that investors with increasing sophistication acquire tranches with decreasing seniority. Mitchell (2004) reviews the financial literature which justifies the creation of structured assets such as CDOs.

4.2 Pricing

In order to simplify the presentation, we introduce two further assumptions:

Assumption 7. Complete market and absence of arbitrage opportunities.

For our purposes we shall use the class of equivalent probability measures $\mathbf{Q}$, where non-dividend paying asset processes discounted using the default-free short rate are martingales. Absence of arbitrage is a necessary requirement for the existence of (at least) one equivalent probability measure, and the assumption of market completeness guarantees its uniqueness. Such an equivalent measure is called a *risk neutral measure* and will be used to derive bonds and CDS pricing formulas.[8]

Assumption 8. Independence of the firms' credit risk and the default-free interest rates under the risk neutral probability measure.

Although the model could accommodate correlation between interest rates and survival probabilities, such an assumption would add a higher degree of complexity into the model because a process would have to be estimated for the default-free short rate as well.[9]

Pricing a CDO consists on finding the appropriate premium s_j for each tranche j. The premium s_j is fixed in such a way that the net present value of the cash flows received/paid by its holders is zero, which implies that, as in a swap or CDS, there is no payment up-front.[10]

Similar to a plain vanilla interest rate swap or a CDS, a CDO consists of two legs: a fixed and a floating leg.[11] The fixed leg represents the payments tranche holders receive (positive cash flows), whereas the floating leg represents the payments they pay (negative cash flows). Consider a CDO with payment dates $\{t_1, ..., t_K\}$, maturity t_K, and notional M, where $\eta = t_{k+1} - t_k$ for all $k = 0, ..., K$. The contract starts at time $t_0 = 0$, and the first premium is due at t_1.

At each payment date t_k the holders of tranche j receive (33), for $k = 1, ..., K$.

[8] See Elizalde (2005b, Appendix A) for an analysis of the different scenarios under which the transition from the physical to the equivalent (or risk neutral) probability measure can be accomplished.

[9] Elizalde (2005d) estimates a reduced form model in which default probabilities depend on default-free interest rates. The results show that the effect of default-free rates on default probabilities is small. Moreover, different empirical papers find different signs for that effect.

[10] Although, as we mention below, the equity tranche does not usually follow this convention, it can be priced using similar arguments.

[11] Elizalde (2005a) describes a CDS pricing model in a similar framework.

Thus, the value of the fixed leg at time t_0, denoted by $X_{F,j}$, is equal to

$$X_{F,j} = \sum_{k=1}^{K} \beta\left(t_0, t_k\right) s_j \eta E\left[\left(K_{U_j} - K_{L_j} - Z_{j,t_k}\right) M\right], \tag{36}$$

where $\beta\left(t_0, t_k\right)$ is the discount factor from t_0 to t_k.

At each payment date t_k the holders of tranche j pay (34), for $k = 1, ..., K$. Thus, the value of the floating leg at time t_0, denoted by $X_{V,j}$, is equal to

$$X_{V,j} = \sum_{k=1}^{K} \beta\left(t_0, t_k\right) E\left[\left(Z_{j,t_k} - Z_{j,t_{k-1}}\right) M\right]. \tag{37}$$

The premium s_j is chosen in such a way that

$$X_{F,j} = X_{V,j}, \tag{38}$$

which implies

$$s_j = \frac{\sum_{k=1}^{K} \beta\left(t_0, t_k\right)\left(E\left[Z_{j,t_k}\right] - E\left[Z_{j,t_{k-1}}\right]\right)}{\sum_{k=1}^{K} \beta\left(t_0, t_k\right) \eta\left(K_{U_j} - K_{L_j} - E\left[Z_{j,t_k}\right]\right)}. \tag{39}$$

Z_{j,t_k}, given by (28), is the accumulated loss suffered by the holders of tranche j from the origination of the CDO up to time t_k, expressed as a percentage of the portfolio notional value M.

Given the attachment points K_{U_j} and K_{L_j}, the payment dates $t_1, ..., t_k, ..., t_K$ and the discount factors $\beta\left(\cdot, \cdot\right)$, we need to evaluate the expectations appearing in (39) in order to compute the tranche premium s_j. In particular, we need to evaluate $E\left[Z_{j,t_k}\right]$:[12]

$$E\left[Z_{j,t_k}\right] = E\left[\min\left\{Z_{t_k}, K_{U_j}\right\} - \min\left\{Z_{t_k}, K_{L_j}\right\}\right], \tag{40}$$

for $k = 1, ..., K$.

The characteristics (size, number of firms, default probability of each firm, default correlations between firms, loss given default, ...) of the portfolio will determine the

[12] Note that $E\left[Z_{j,t_0}\right] = 0$ for all tranches j.

distribution function for Z_{t_k} and, as a consequence, the tranche premiums. Note that the percentage losses of the portfolio at each time t_k is a random variable different from the percentage losses of the portfolio at any other time different from t_k. Therefore, we would need the distribution functions for the random variables $Z_{t_1}, ..., Z_{t_K}$, i.e. for each payment date $t_1, ..., t_K$.

The issues in this section are also covered by Hull and White (2004), Willemann (2004), Amato and Gyntelberg (2005), De Prisco, Iscoe and Kreinin (2005), and Hull, Pedrescu and White (2005). Bluhm (2003) reviews the main CDO modelling techniques and offers illustrating examples and applications. The Committee on the Global Financial System (2005) reviews the role of rating agencies in the pricing and rating of structured products, mainly CDOs. It also provides the main characteristics of the CDO pricing models used by the main rating agencies.

4.3 Vasicek model: homogeneous large portfolio

As shown in Section 3, if the CDO underlying portfolio satisfies Assumptions 1 to 6, the percentage losses of the portfolio at each time t, Z_t, is given by the loss given default λ_t, similar for all firms in the portfolio, times the portfolio default rate Ω_t.

The premium s_j of a tranche with attachment points K_{U_j} and K_{L_j} is given by (39). Using (25) and (28), $E\left[Z_{j,t_k}\right]$ can be expressed as

$$E\left[Z_{j,t_k}\right] = \int_0^1 \left(\min\left\{\lambda_{t_k}\omega, K_{U_j}\right\} - \min\left\{\lambda_{t_k}\omega, K_{L_j}\right\}\right) dF\left(\omega; p_{t_k}, \rho_{t_k}\right), \tag{41}$$

for $k = 1, ..., K$. The integral in (41) needs to be numerically evaluated.

The premium s_j of tranche j depends, in the Vasicek model presented, on:

- Upper and lower attachment points K_{U_j} and K_{L_j}.
- Term structure of interest rates at time t_0, given by the discount factors $\beta\left(t_0, t_1\right)$, ... , $\beta\left(t_0, t_K\right)$.
- Frequency of payments η.

- Default probabilities $p_{t_1}, ..., p_{t_K}$ (assumed equal for all credits in the portfolio).

- Default correlations $\rho_{t_1}, ..., \rho_{t_K}$ (assumed equal for all credits in the portfolio).

- Losses given default $\lambda_{t_1}, ..., \lambda_{t_K}$ (assumed equal for all credits in the portfolio).

- Distribution function $F\left(\cdot; p_{t_k}, \rho_{t_k}\right)$ of the underlying portfolio default rate.

It is common practice to assume the correlation parameter to be the same not only across firms, but also for all time horizons:

$$\rho_{t_k} = \rho, \quad \text{for all } k = 1, ..., K. \tag{42}$$

Moreover, ρ is usually estimated (where usually means *usually*, not *adequately*) from correlations of equity returns, typically ranging from 0 to 30 percent. To estimate the default probabilities p_{t_k} the most straightforward option is to use a reduced form model calibrated from bond or CDS prices. In the same way, it is usual practice to assume λ_t constant across time: λ.

With respect to the use of a single correlation parameter for all firms, St. Pierre et al. (2004, p. 7) state that "Although some market participants have attempted to use information from equity returns or spread changes to estimate a correlation value for every pair of names, the market continues to use a single correlation for the entire portfolio primarily because a clearly compelling alternative has not yet emerged."

The issues in this section are also covered by Gregory and Laurent (2003) and Amato and Gyntelberg (2005).

4.4 Sensitivity of tranche premiums to the model parameters

Clearly, tranche premiums are a positive function of the firms' default probability and of the losses in case of default. The impact of the default correlation on tranche premiums is slightly more complicated. For the equity tranche, a higher default correlation increases the probability that no defaults will occur and therefore decreases the premium. The opposite is true for the most senior tranche, where a high default correlation increases the probability that of a high number of defaults occurring, increasing its expected losses and therefore its premium. While equity and senior tranche premiums are monotonic on the default correlation, its impact on mezzanine tranches is not clear cut.

Duffie and Garleanu (2001), Gibson (2004), Hull and White (2004), Willemann (2004), and Amato and Gyntelberg (2005) present an analysis of the sensitivity of tranche premiums to the model parameters, in particular to the correlation coefficient. Gregory and Laurent (2004) describe in detail CDO sensitivities with respect to the correlation parameter.

4.5 Trading issues

CDOs can either be constructed using a portfolio of loans or bonds (cash CDOs) or using a portfolio of CDS (synthetic CDOs). Figure (1) shows the notional of cash and synthetic CDOs issued and sold to investors in the last 9 years. Although cash CDOs were more popular than synthetic CDOs in the second half of the 90's, synthetic CDOs have taken the lead since 2000.

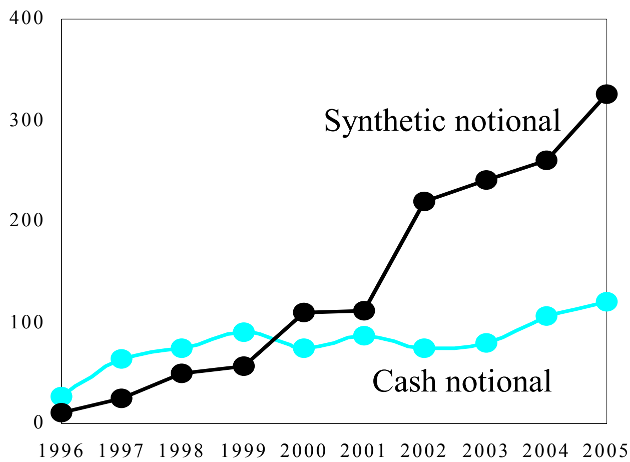

Figure 1: CDO notional (billion USD). Source: Financial Times (2005a).

In recent years, a new way of selling and buying CDO tranches has became extremely popular, mainly due to the appearance of standardized markets: *single CDO tranches*.[13] As explained by Hull, Pedrescu and White (2005, p. 5), "Standard portfolios and standard tranches are defined. One party to a contract agrees to buy protection on an individual tranche; the other party agrees to sell protection on the tranche. Cash flows are circulated in the same way as they would be if a synthetic CDO were constructed for the portfolio. However, in single tranche trading, the underlying portfolio of credits is never created. It is merely a reference portfolio used to calculate cash flows."

This market works somewhat similarly as the options and futures markets on stock indexes such as Dow Jones, S&P, FTSE, ... First, a portfolio or *index* of firms is formed. As in the case of stock indexes, the composition of the index is publicly known, chosen by the organizing market and reviewed and updated periodically according to the liquidity and traded volume of the firms.[14] Second, CDO tranches are traded using the firms in the index as if they were in the underlying portfolio of the CDO.

Dow Jones administers the main credit derivatives indexes which are used as underlying portfolios for single CDO tranche trading. There exist indexes for the US, emerging markets, Asia, Europe, ... as well as indexes for different sectors of some of those areas.[15]

The two most traded indexes are the *Dow Jones CDX NA IG* and the *Dow Jones iTraxx Europe*, which are composed of 125 equally weighted investment grade US and European firms respectively.

The portfolios underlying the previous indexes are used to construct tranches in the same way as the portfolios underlying a CDO. A tranche defined on any of these indexes has a well defined cash flow structure and, as such, it can be traded among

[13] Hull and White (2004), Amato and Gyntelberg (2005), and Hull, Pedrescu and White (2005) provide alternative descriptions of these markets.

[14] Since the index serves as an underlying portfolio to trade with the firms' credit risk, the liquidity and volume of trade refer to the firms' defaultable instruments such as bonds and CDS.

[15] For more information see `www.djindexes.com`.

market participants.[16] Credit derivatives dealers provide bid and ask quotes for the premium of each tranche.

Once an investor enters into a (long or short) position in a CDO tranche for a given amount of money, the evolution (i.e. default status) of the firms in the index during the specified maturity will determine the corresponding payoffs. The most common maturity is 5 years.

When a firm in the index defaults, it is taken off from the index and a new one is introduced. This new firm is taken into account for new, but not for previously contracted, deals. Changes in the composition of the index do not necessarily involve defaults, but also liquidity and related considerations.[17]

The theoretical formula for the premium (39) characterizing a CDO tranche was derived assuming there was no up-front payment. The way in which single CDO tranches are quoted follows that assumption except for the equity tranche. In the equity tranche, the protection buyer, i.e. tranche holder, receives a periodic premium of 5% of the outstanding notional, as well as an up-front payment, which is what is quoted in the market.

Andersen, Basu and Sidenius (2003), Gibson (2004), and St. Pierre et al. (2004) analyze the dynamic hedging of short positions on single-tranche CDOs using CDSs of the firms underlying the CDO portfolio.

Finally, for those worried about the tax treatment of CDO returns, Bloomfield and Shamrakov (2005) offer some tips to maximize tax benefits.

4.5.1 Moral hazard

It is often the case in cash CDOs that the originator of the CDO, usually a bank repackaging and selling credits in its portfolio, retains the most risky (equity) tranche. As Duffie and Garleanu (2001) and Gibson (2004) point out this responds mainly to two facts. First, the bank knows better than anybody else the quality of the credits

[16] The structure of the upper and lower attachment points can vary between indexes.

[17] See Dow Jones Indexes (2005).

underlying the CDO and, in order to ensure prospective investors that they won't be cheated, it retains the riskiest tranche as a signal of the fairness of the deal. This asymmetry of information is more acute when the underlying portfolio is made up of non-traded instruments, such as bank loans. Second, in most cases the bank is the one in charge of monitoring the service of the credits, which can influence the probability of those credits making the scheduled payments and therefore the cash flows of the CDO. Retaining the equity tranche keeps the incentives of the originating bank to keep to monitor the credits.

Franke and Krahnen (2005) provide a review of theoretical literature showing that first loss positions are optimal arrangements for the originators of CDOs and similar structured assets. They also mention research conducted by the Deutsche Bank empirically verifying that major German banks do actually hold significant positions in the equity tranches they originate.

The explosion of CDO trading in recent years means that banks have been able to get rid of a huge amount of unwanted credit risk. But, according to the previous arguments, they have kept a high fraction of equity tranches, the riskiest ones. As a consequence, they have reduced their exposure to high credit risk losses but they have increased the fraction of their credit risk coming from first losses. Financial Times (2005d) documents that this "has left some issuing banks with large unwanted trading positions. This year, banks have been adopting an array of techniques aimed at either reducing this risk overhang or making the higher-yielding, riskier credit pieces more appealing to a broader range of investors." The article mentions a survey by Fitch Ratings showing evidence of the high exposure to issuing banks to CDO equity tranches and their efforts to reduce this exposure to avoid, for example, the high capital charges which these positions will generate under Basel II.

4.5.2 Types of CDOs

This section borrows directly from Fitch (2004b), which presents a detailed analysis of CDO mechanics as well as Fitch's methodology to price, analyze and assign ratings to CDOs.

Fitch (2004b) considers three criteria to classify CDOs: assets being securitised, motivation behind the securitization, and way in which the CDO transfers the credit risk of the underlying portfolio. According to the assets being securitised, we can mainly distinguish between bonds (*CBO*s) and loans (*CLO*s). The reasons behind the issuance of the CDO determines whether it is an *arbitrage* CDO or a *balance sheet* CDO. Balance sheet CDOs are those created by financial institutions to transfer part of the credit risk in their balance sheet to other investors. Arbitrage CDOs are those issued in order for the issuer to (p. 2) "profit on the margin between the weighted average return received on a portfolio of debt obligations and the cost of hedging the risk in the capital markets via the issuance of the CDO notes or swaps." Finally, and as we have already mentioned, a CDO can either be a *cash flow* CDO, where the issuer actually owns the underlying portfolio, or a *synthetic* CDO, whose underlying portfolio is not owned by the issuer but it is based on an index of firms.[18]

4.5.3 Correlation smile and base correlations

We all know what "implied volatility" means in European option pricing. Using the popular Black-Scholes formula for valuing options, the option price is a closed function of the short rate, time to maturity, stock price, and volatility of the stock price. For a given market price of the option and using the short rate, time to maturity, and stock price, we can back up, using the Black-Scholes formula, the volatility level which yields an option price equal to the price quoted in the market. Such a volatility level is called implicit volatility.

The same exercise can be performed for the prices of CDO tranches. The market price of a single tranche CDO would play the role of the option market price; the Vasicek asymptotic single risk factor formula (39) would play the role of the Black-Scholes formula, and the default correlation coefficient ρ (assumed constant across

[18] Fitch (2004b) also distinguishes between *static* and *revolving* CDOs (p. 12), as well as between underlying portfolios with bullet maturities or amortizing principal schedules (p. 14).

firms and time) that of the stock price volatility.

Therefore, given the market price of a single CDO one can, using the pricing formula (39) and fixing the value of the rest of the parameters, (numerically) compute the correlation coefficient ρ which matches such a price.[19]

Using market prices of single CDO tranches with the same underlying portfolio, we can compute the *implied default correlation* for each tranche. If the model behind the pricing formula (39) were correct we should obtain the same correlation for all tranches. However, that is not the case. In general, the implied correlation is higher for equity and most senior tranches than for mezzanine tranches, which is known as the *correlation smile*.

Amato and Gyntelberg (2005, p. 84) present several possible explanations for the correlation smile: (i) there is segmentation among investors across tranches and these different investor groups hold different views about correlations; (ii) the smile reflects market participants' uncertainty about how best to model credit risk correlations (the implication is that equity tranches, which are more sensitive to correlations, contain a model risk premium embedded in their prices); (iii) local demand conditions in prices; and (iv) that market participants might be using other pricing models.

The first explanation offered by Amato and Gyntelberg finds support on the desire of issuing banks to get rid (ex-post) of all their positions in equity tranches they kept when issuing CDOs; see discussion in Section 4.5.1. As explained there, banks kept positions in the equity tranches of the CDOs they issued to avoid problems of moral hazard. But these positions have become large enough as for them to try to pass them into the market by re-selling them with new names and some added guarantees. Financial Times (2005d) argue that "This demand imbalance between different sections of the CDO capital structure - also referred to as correlation book - has pushed up equity tranche spreads, which, according to some bankers, means they offer higher returns and better value than warranted by fundamental factors. The

[19] Note that this can be done with any CDO pricing model. However, the Vasicek asymptotic single factor model has become the market standard.

spreads and returns on mezzanine tranches, meanwhile, have been suppressed."

As pointed out by Hull and White (2004), "tranche implied correlations must be interpreted with care. For the equity tranche (the most risky tranche in a CDO, typically 0% to 3% of the notional) higher implied correlation means lower value to someone buying protection. For the mezzanine tranche ... the value of the tranche is not particularly sensitive to correlation and the relationship between correlation and breakeven spread, ..., may not be monotonic. For other tranches higher implied correlation means higher value to someone buying protection."

Apart from interpretation problems, implied correlations suffer another two major problems, pointed out by, among others, McGinty and Ahluwalia (2004), Willemann (2004), and van der Voort (2005). First, since mezzanine tranche premiums are not monotonic in the correlation, implied correlations might not be unique. Second, implied correlations depend on the upper and lower tranche attachment points and, as a consequence, they can not be interpolated to price other tranches with different attachment points.

McGinty and Ahluwalia (2004) came up with a clever way of solving such problems: *base correlations*. They make use of the monotonicity of the equity tranche on the default correlation, and construct fictitious equity tranches which are then used to construct mezzanine tranches. However, base correlations are not bullet proof as shown by Willemann (2004), and, as pointed out by Hull and White (2004), they are even more difficult to interpret than implied correlations.

For a discussion and a list of references on base correlations see Finger (2004). St. Pierre et al. (2004) explain how to derive base correlations from market observed tranche premiums and how to employ them to value non-standard tranches.

4.5.4 Exotic CDOs

Besides its importance in terms of traded volume, multiname credit derivatives and CDOs in particular, are constantly evolving in terms of new products, new markets, new trading conventions, ... Such developments are the forces behind the latest advances in modelling and calibration techniques. As an example of such new products, the so-called CDO^2 is a CDO of CDOs: a portfolio of CDOs is assembled, tranched, and sold to investors. See Baheti et al. (2005) and Fitch (2004a) for a description of CDO^2.

Other *exotic* CDO products include forward starting CDOs, options on CDO tranches, leverage super senior CDOs, and bespoke CDOs. Most of this derivatives have payoff structures linked to the evolution of tranche premiums (or their pricing requires to model such evolution). Thus, their pricing and hedging demand models which take into account the dynamics of tranche premiums. The use of the Vasicek single factor model (and its extensions) is limited for this type of products, because they are (so far) static pricing models: they are able to produce tranche premiums for a given set of parameter models which is valid for one particular day, but not to consistently provide a model for the dynamics of such prices.

The dynamics needed for pricing these exotic CDOs can be obtained, for example, with intensity models (see Introduction). Additionally, a new set of dynamic credit portfolio models has been put forward by Andersen, Piterbarg and Sidenius (2005), Bennani (2005), and Schönbucher (2005). The authors present alternative proposals to model directly the dynamics for the portfolio losses.

4.6 Extensions of the Vasicek model

The attractiveness of the Vasicek model for CDO pricing lies in its simplicity and relatively straightforward application, which only requires to compute a set of numerical integrals. However, the assumptions about the characteristics of the underlying portfolio are strong.

There exist quite a broad literature which, building on the previous model, relaxes one or several assumptions and comes up with another pricing model. In some, although not in all, cases this implies the use of Monte Carlo simulation. The popularity of these models is confirmed by its use by the main rating agencies.[20]

[20] See the Committee on the Global Financial System (2005, p. 18).

In what follows we provide brief a review of this literature, without going in detail into the mechanics of each pricing model.

4.6.1 Homogeneous finite portfolio

CDO underlying portfolios vary in size. For example, the two most popular single tranche CDOs, Dow Jones CDX NA IG and Dow Jones iTraxx Europe indexes, are composed of 125 reference entities, with an *equal weighting* given to each.

If we relax Assumption 4 (the number of credits in the portfolio is very large, $N \to \infty$), one does not get the very nice equation (25) for the distribution function of the portfolio losses.

If Assumptions 1-3 are satisfied, the number of defaults in the portfolio at time t, denoted by L_t, is, conditional on the common factor Y_t, a binomial $(N, p(Y_t))$ random variable.

As a consequence, the conditional probability of the portfolio experiencing x defaults before t is given by

$$\mathbf{P}(L_t = x \mid Y_t) = \begin{pmatrix} N \\ x \end{pmatrix} p(Y_t)^x (1 - p(Y_t))^{N-x}. \tag{43}$$

Since the common factor Y_t follows an i.i.d. standard normal random variable, the unconditional probability of the portfolio experiencing x defaults before t is given by

$$\mathbf{P}(L_t = x) = \int_{-\infty}^{+\infty} \mathbf{P}(L_t = x \mid s)\, d\Phi(s), \tag{44}$$

where $\Phi(\cdot)$ is the distribution function of a standard normal random variable.

If Assumption 6 holds, i.e. the size of each credit in the portfolio is similar, each default generates a loss of $\frac{1}{N}$ percent of the total portfolio initial value M. Therefore, the distribution function of the portfolio percentage losses at time t is

$$\hat{F}(\omega; p_t, \rho_t) = \sum_{x=0}^{N} \mathbf{P}(L_t = x)\, \mathbf{1}_{\left\{\omega \geq \frac{x}{N}\right\}}, \tag{45}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

Keeping Assumption 5 (similar loss given defaults across firms), the premium s_j of a tranche with attachment points K_{U_j} and K_{L_j} is still given by (39), but with $E\left[Z_{j,t_k}\right]$ given by

$$E\left[Z_{j,t_k}\right] = \int_0^1 \left(\min\left\{\lambda_{t_k} x, K_{U_j}\right\} - \min\left\{\lambda_{t_k} x, K_{L_j}\right\}\right) d\hat{F}\left(x; p_{t_k}, \rho\right), \tag{46}$$

for $k = 1, ..., K$, instead of (41).

Hull and White (2004), among others, offer a CDO pricing model for finite portfolios with two different implementation approaches.

4.6.2 General distribution functions

In the Vasicek single risk factor model, a firm n defaults at time t if the random variable $X_{n,t}$ is lower than a default threshold K_n. Moreover, $X_{n,t}$ is a function (11) of a common or systematic factor Y_t and a firm idiosyncratic factor $\varepsilon_{n,t}$, assumed i.i.d. standard normal random variables. As a consequence, the credit risk dependence structure among firms is given by a normal multivariate distribution or normal copula.

However, as argued by Frey, McNeil and Nyfeler (2001), there is no compelling reason for choosing normal random variables for the distributions of Y_t and $\varepsilon_{1,t}, ..., \varepsilon_{N,t}$ and, as a consequence, of $X_{1,t}, ..., X_{N,t}$. They show, in the framework of the single factor model, that the aggregate portfolio distribution is extremely sensitive to the exact nature of the multivariate distribution of the latent variables $X_{1,t}, ..., X_{N,t}$. Normal mixture distributions, t-student, and generalized hyperbolic distributions, comprise a useful source of alternative models for the latent variables.[21]

See Schönbucher (2000, 2002), Hull and White (2004), and Burtschell, Gregory and Laurent (2005) for extensions of the Vasicek single factor model with general distribution functions for the systematic and idiosyncratic factors. Hull and White (2004) consider normal and t-student distributions, while Burtschell, Gregory and Laurent (2005) also consider several copula models. Schönbucher (2002) analyzes

[21] Mencia and Sentana (2004) analyze the properties of the generalized hyperbolic distribution, which has the normal and t-student distribution as special cases. They also propose statistics to test for normality and t-student.

the class of Archimedean copulas, and Andersen, Basu and Sidenius (2003) use a t-student copula. Daul et al. (2003) generalize the t-copula to model large sets of risk factors of different classes, allowing to more accurately estimate the tail dependence present in the data.

One of the drawbacks of the normal copula is, as we mentioned previously, that it cannot fit the prices of the different CDO tranches with a single correlation coefficient, generating a correlation smile. As Kalemanova, Schmid and Werner (2005) explain, researchers blame the lack of tail dependence in the normal copula and propose copulas with positive tail dependence such as t-student, ... These distributions correct the pricing performance of the model, but at the cost of a much higher computation time. The authors propose the use of a Normal Inverse Gaussian (NIG) distribution, a special case of the generalized hyperbolic distribution, which is shown to improve both the computation time and the pricing accuracy of the model. See also Guegan and Houdain (2005) for an application of the NIG distribution to CDO pricing.

4.6.3 Heterogeneous finite portfolio

The assumption of a finite portfolio comprised of heterogeneous credits represents the most real case. In an heterogeneous portfolio, each credit has a different default probability $p_{n,t}$, loss given default $\lambda_{n,t}$ and exposure to the systematic risk factor $\rho_{n,t}$. Each credit n represents a fraction f_n of the initial portfolio value.

There are several techniques to derive the distribution of the portfolio loss rate Z_t. We present here one based on the Fast Fourier Transform (FFT), which is probably the most intuitive, although it has been shown not to be the fastest among the available alternatives (cf. Andersen, Basu and Sidenius 2003, Gregory and Laurent 2003, De Prisco, Iscoe and Kreinin 2005, and Hull and White 2004.)[22]

The default probability of firm n conditional on the realization of the common

[22] De Prisco, Iscoe and Kreinin (2005) present a review and comparison of these models.

factor becomes

$$p_n(Y_t) = \Phi\left(\frac{\Phi^{-1}(p_{n,t}) - \sqrt{\rho_{n,t}}Y_t}{\sqrt{1-\rho_{n,t}}}\right). \tag{47}$$

The percentage portfolio loss at time t, Z_t, is given by

$$Z_t = \sum_{n=1}^{N} L_{n,t} f_n \lambda_{n,t}, \tag{48}$$

where, for each firm n, the random variable $L_{n,t}$ takes value 1 if the firm has defaulted up to time t and 0 otherwise.

The characteristic function of the random variable Z_t conditional on the common factor is

$$\begin{aligned}
\Psi(u \mid Y_t) &= E\left[e^{iuZ_t} \mid Y_t\right] && (49)\\
&= E\left[e^{iu\sum_{n=1}^{N} L_{n,t} f_n \lambda_{n,t}} \mid Y_t\right] && (50)\\
&= E\left[\prod_{n=1}^{N} e^{iuL_{n,t} f_n \lambda_{n,t}} \mid Y_t\right], && (51)
\end{aligned}$$

where $i = \sqrt{-1}$.

Conditional on the common factor Y_t, the random variables $L_{1,t}, ..., L_{N,t}$ are independent Bernoulli variables:

$$\begin{aligned}
\Psi(u \mid Y_t) &= \prod_{n=1}^{N} E\left[e^{iuL_{n,t} f_n \lambda_{n,t}} \mid Y_t\right] && (52)\\
&= \prod_{n=1}^{N} \left[p_n(Y_t)\, e^{iuf_n\lambda_{n,t}} + (1 - p_n(Y_t))\right] && (53)\\
&= \prod_{n=1}^{N} \left[1 + p_n(Y_t)\left(e^{iuf_n\lambda_{n,t}} - 1\right)\right]. && (54)
\end{aligned}$$

Integrating over Y_t we compute the unconditional characteristic function of the percentage portfolio loss

$$\Psi(u) = \int_{-\infty}^{\infty} \Psi(u \mid Y_t)\, d\Phi(Y_t). \tag{55}$$

Once we have numerically computed the unconditional characteristic function of the percentage portfolio loss Z_t we can use the FFT to recover its distribution function which then is employed in the CDO pricing formulas.

As Burtschell, Gregory and Laurent (2005) point out, the FFT (or any alternative technique) can be used independently of the underlying model, as long as we can derive the conditional default probability $p_n(Y_t)$.

Additional papers which price CDOs with heterogeneous finite underlying portfolios include Andersen and Sidenius (2004) and Burtschell, Gregory and Laurent (2005).

4.6.4 Stochastic default correlations

The most usual practice is to consider default correlations constant through time, similar across firms, and independent of the firms' default probabilities. Regarding the latter issue, Hull, Pedrescu and White (2005) argue that there is a growing body of empirical research suggesting that correlations are positively dependent on default probabilities.[23] The authors use a first passage model, similar to the Merton (1974) structural model in which the Vasicek factor model is based, but which allows firms to default at any point in time. Default correlations are made stochastic and correlated with the systematic factor, generating a better fit to CDO market data than the basic model with constant default correlation. The model can also be extended to include more than one systematic factor and is implemented through Monte Carlo simulation.

Burtschell, Gregory and Laurent (2005) consider the case where the random variables $X_{1,t}, ..., X_{N,t}$ are given by

$$X_{n,t} = \sqrt{\rho_n} Y_t + \sqrt{1 - \rho_n} \varepsilon_{n,t}, \tag{56}$$

for all $n = 1, ..., N$, where $\rho_1, ...\rho_N$ are independent stochastic correlations. Unlike Hull, Pedrescu and White (2005), the default correlations are independent of the

[23] The authors refer to De Servigny and Renault (2002), Ang and Chen (2002), and Das, Freed and Kapadia (2004).

systematic risk factor. The authors show that stochastic correlation practically eliminates the correlation smile for, among others, normal and t-student distributions for the systematic and idiosyncratic factors.

Andersen and Sidenius (2004) also consider a (multifactor) model with stochastic default correlation, allowing default correlations to be higher in bear markets than in bull markets.

4.6.5 Multifactor models

Is it one systematic risk factor enough to capture all credit risk correlation across firms? A proper answer to this question is still lacked. Wilson (1998) and Elizalde (2005d) represent attempts to answer it, from two different angles.

In a multifactor model, the random variable $X_{n,t}$ determining whether firm n defaults or not at time t, depends on various, rather than one as in the Vasicek single factor model (11), common factors:

$$X_{n,t} = \sqrt{\rho_1} Y_{1,t} + ...\sqrt{\rho_J} Y_{J,t} + \sqrt{1 - \rho_1 - ... - \rho_J}\varepsilon_{n,t}, \tag{57}$$

where $Y_{j,t}$, for $j = 1, ..., J$, are different common factors and $\varepsilon_{n,t}$ is a firm idiosyncratic factor. The firms' default correlation structure is given by their dependence on the common factors through the coefficients $\rho_1, ..., \rho_J$. Again, as in the Vasicek single factor model, one can assume those coefficients to be different for each firm n and across time.

Schönbucher (2000) and Hull and White (2004) deal with multifactor models.

4.6.6 Random loss given default

Andersen and Sidenius (2004) and Hull and White (2004) consider the case of random loss given defaults, correlated with default probabilities.

4.6.7 Totally external defaults

In a recent paper, van der Voort (2005) addresses, what he considers, a fundamental problem of the standard one factor Gaussian, i.e. Vasicek single risk factor, pricing methodology. In such a model, a default event will strongly affect the available information on the common factor, significantly increasing the default probability of the rest of the firms in the underlying portfolio. The author argues (p. 5) that this "fundamental shortcoming of the model is caused by the fact that the model is not able to explain defaults which are not caused by macro economic behavior, but are entirely external." He proposes to introduce an extra idiosyncratic factor in the model, to account for *totally external default risk*, caused by fraud, legal issues, ... (e.g. Enron, Worldcom, Parmalat, ...) This way, totally external defaults do not affect the available information about the systematic risk factor.

After deriving formulas for the default probabilities and portfolio loss rate, the author shows how this extended model can account for the implied correlation smile.

4.6.8 Focusing on a single tranche

All the models analyzed so far start deriving the distribution function for the losses in the total portfolio Z_t, which is then used to derive the losses in each single tranche j, $Z_{j,t}$ (given by 28). However, the seller of protection, i.e. tranche holder, of a given tranche j (and the buyer of protection in single tranche CDOs) is only interested in the distribution of $Z_{j,t}$. De Prisco, Iscoe and Kreinin (2005) develop a new analytical approach, building upon the Vasicek single factor model for CDO pricing, for directly computing the loss distribution of a single tranche Z. They use a compound Poisson approximation and show that this technique is as accurate of competing models such as Monte Carlo methods which first compute the total distribution of the portfolio. Moreover, when we are just interested in valuing a single tranche, their technique is faster in most cases.

Other extensions have been considered, most of them included in the papers mentioned during the text. Finger (2004) presents an alternative survey of different variations of the standard model for CDO pricing.

4.7 Parameter calibration

Choosing a particular model from the ones presented above is already a difficult task, which trades-off model simplicity versus the validity of the assumptions. But that is only half of the problem. Once a model is specified, one has to feed it with values for the different parameters in order to obtain prices for CDO tranches.

There are three main groups of parameters which have to be estimated: marginal default probabilities, default correlations and losses given default.

4.7.1 Default probabilities

Marginal default probabilities can be obtained in several ways: (i) using an structural model (similar or more complicated than the Merton 1974 model) and data of the firms' assets and liability structure, (ii) using a reduced form model and data of the firms' defaultable instruments (bonds, credit default swaps, ...) market prices,[24] (iii) using information from rating agencies about default probabilities, ...

4.7.2 Loss given default

Regarding losses given default, the convention is to use historical (generally published by rating agencies) data to select them, depending on the seniority of the claims analyzed and assuming they are constant across time and across seniorities.[25] The Committee on the Global Financial System (2005) reviews the main characteristics of the CDO pricing models used by the main rating agencies. With respect to recovery rate assumptions, the Committee (p. 18) states that they "continue not to conform fully with mounting empirical evidence of substantial cyclical variability in recoveries and negative correlation with default probabilities."[26] However, it also recognizes (p. 44) that "more emphasis is also being put on systematic variation in recovery rates" within rating agencies CDO pricing models.

[24] See Galiani (2003) for an application using CDSs, and Li (1998) and Elizalde (2005a) for applications using bond prices.

[25] See Moody's (2000) and Altman and Arman (2002).

[26] See Covitz and Han (2004) and Altman et al. (2005).

The Basel Committee of Banking Supervision set up a *Loss Given Default Working Group* in September 2004 in order to analyze the relationships between losses given default and economic conditions (which can also affect default probabilities and default correlations) and the treatment of losses given default in the banking industry. Their main findings, included in BCBS (2005), emphasize that losses given default are lower than average during times of high default rates, that data limitations pose an important challenge to estimate such relationships, and that there is currently little consensus (across banks) with respect to the way of incorporating the correlation between losses given default and default probabilities in the calibration of their credit models.

Fitch (2004b, p. 8) explains the differences in portfolio losses depending on whether the CDO is a cash flow or a synthetic CDO, which make a big difference when calibrating losses given default. "In a cash flow CDO, recoveries are always achieved by either selling the defaulted asset or going through the work-out process. In a synthetic CDO, losses and recoveries are determined by either cash or physical settlement. Under a cash settlement, a protection payment is based on the difference between the par value of an obligation selected for valuation and its post-credit-event market value determined in a bidding process, ... Under physical settlement, the protection buyer is paid the par amount of the defaulted obligation and must deliver an obligation to the CDO issuer." Therefore, under physical settlement, one has to value the option that the protection buyer has to choose between the different (probably pre-specified in the contract) obligations of the defaulted firm.

4.7.3 Correlation

Without doubt the calibration of default correlation is the most complicated part of the process. According to Duffie and Garleanu (2001, p. 3), "Currently the weakest link in the chain of CDO analysis is the availability of empirical data that would bear on the correlation, actual or risk neutral, of default." Schönbucher (2003, p. 289) agrees that "Default correlation and default dependency modelling is probably the most interesting and also the most demanding open problem in the pricing of credit derivatives."

Although the way in which default correlation is introduced in the above models is quite convenient in terms of modelling, it is quite inconvenient in terms of calibration. The exposure to the systematic risk factor ρ represents the impact of the (unobservable) systematic risk factor on a random variable ($X_{n,t}$) which is a non-linear transformation of the firm's asset value (which itself is an unobserved and complicated process to estimate). The major problem when calibrating ρ is not the availability of data sources, but the difficulty of interpreting what ρ represents.

The usual practice is to use equity return correlations directly (e.g. Fitch, 2004b, p.10). But what is the relationship (in the model) between equity returns and the exposure to the systematic risk factor ρ? Whatever it is (and this depends on the underlying model), they are probably not the same. De Servigny and Renault (2002) and Zeng and Zhang (2002) conclude that using equity correlation as a proxy of the model default correlation is clearly insufficient.

The Committee on the Global Financial System (2005, p. 44) reports that, among the major rating agencies, Moody's and Fitch use assumptions based on equity returns correlations. In contrast, Standard and Poor's calibrates ρ to historically observed default correlations (which seems a more appropriate solution).

Daniels, Koopman and Lucas (2005), after reviewing different approaches for modelling default dependencies using observable macroeconomic variables as a systematic risk factor, present a model to decompose default risk into two unobservable factors: a systematic and idiosyncratic factor. Such a model is similar to the standard Vasicek single risk factor model. Their estimation methodology allows for a dynamic structure for the systematic risk factor.

The problem of calibrating the exposure to the systematic risk factor is a second order problem. We must first understand the dynamics driving credit risk correlations, How do they vary across business cycles?, Are they related to default probabilities?, How many factors (economic wide, sectorial, ...) affect them? In which degree?

... Moody's Investor Service (1997), Das, Freed and Kapadia (2002), De Servigny and Renault (2002), and Elizalde (2005d) present answers to those and other related questions.

Finger (2004, p. 122), after reviewing the calibration of default correlations concludes:

> "Do any of the empirical methods of estimating correlation provide reliable results, or is correlation in the standard model purely a technical factor, unrelated to anything that is truly observable? It will be difficult to continue to use the standard model if there is no empirical way to at least estimate a range for the correlation parameter."

As a last caveat, it is surprising (and scaring) how most of the papers cited above applying the Vasicek single risk factor model and its extensions to price CDOs do always present "numerical implementations" to analyze the mechanics of the models, without any mention about how to compute the model parameters, in particular default correlation. Before going on with new and fancier extensions we should be worrying about empirically calibrating and testing existing ones.

Appendix

A Bank capital regulation

The New Basel Accord on Banking Supervision and Regulation, known as Basel II and contained in Basel Committee on Banking Supervision (BCBS, 2004), requires banks to, among other things, hold a minimum level of capital, referred to as regulatory capital. The idea is to align regulatory capital with the risks underlying banks' assets, in such a way that a more risky bank will be required to hold a higher amount of capital.

Subject to the approval of the corresponding supervisor, banks satisfying certain standards of sophistication (in terms of its risk management process, databases, ...) will compute their regulatory capital using the so called internal ratings based (IRB) approach. Under the IRB approach, regulatory capital is computed using a formula, based on Vasicek's asymptotic single factor model.

Under the IRB approach, banks' regulatory capital is the sum of the regulatory capital assigned to each of the following asset classes: corporate, sovereign, bank, retail, and equity, defined in BCBS (2004, paragraphs 215-243). One can think of each of these asset classes as portfolios of similar assets, issued by different firms, and apply the model presented in this report to compute its loss distribution function.

According to IRB approach, bank capital must cover losses due to loan defaults with a probability (or confidence level) of $99,9\%$. In particular, given the distribution function $F(\omega; p_t, \rho_t)$ for the fraction of defaulted loans, let $\widehat{\omega}$ be the critical value such that

$$\Pr(p_t \leq \widehat{\omega}) = F(\widehat{\omega}) = 0.999,$$

which implies $\widehat{\omega} = F^{-1}(0.999; p_t, \rho_t)$.

In the case of a portfolio of corporate, sovereign or bank exposures with an effective maturity of one year $t = 1$, the capital requirement k is given (as a percentage of the portfolio value) by[27]

$$\begin{aligned} k &= \lambda F^{-1}(0.999; p_1, \rho_t) && \text{(A1)} \\ &= \lambda \Phi\left(\frac{\Phi^{-1}(p_1) + \sqrt{\rho_t}\Phi^{-1}(0.999)}{\sqrt{1-\rho_t}}\right), && \text{(A2)} \end{aligned}$$

where:

- λ is the (constant across time) loss given default.[28]

[27] See BCBS (2004, p. 69) for retail exposures and BCBS (2004, p. 72) for equity exposures.

[28] BCBS (2004, p. 62) describes the approaches for deriving an estimate of λ. Under the most simple approach, claims on corporates, banks and sovereigns not secured by recognized collateral will be assigned $\lambda = 45\%$ if they are senior claims and $\lambda = 75\%$ if subordinated. See also BCBS (2005) for a discussion of the relationship between λ and economic conditions and how banks should incorporate it when computing regulatory capital.

- The correlation coefficient ρ_t is assumed to be a decreasing function of the default probability p_t given by

$$\rho_t = 0.24 - 0.12 \frac{1 - e^{-50 p_t}}{1 - e^{-50}}, \tag{A3}$$

and represented in Figure (2). $\rho_t = 0.24$ for $p_t = 0$ and $\rho_t = 0.12$ for $p_t = 1$.[29]

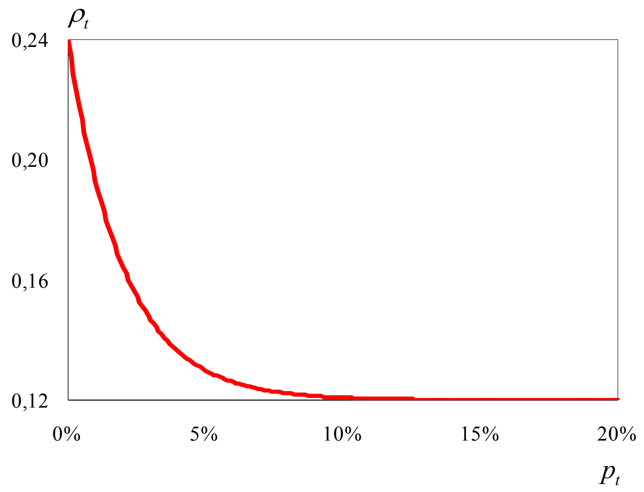

Figure 2: Correlation coefficent ρ_t as a function of the defaut probability p_t.

- If the effective maturity is different than one year, the capital requirement k given by (A1) has to be multiplied by a coefficient δ which is a function of the effective maturity t and the default probability p_t. δ is increasing in the effective maturity for all default probabilities, and is increasing (decreasing) in the default probability for maturities t (lower) higher than one year.[30]

Capital requirements are thus computed as the percentage of the portfolio value the bank needs to set aside in order to be able to cover losses due to loan defaults when no more than 99.9 percent of the loans or credits in the portfolio default.

Figure (3) represents capital requirements (A1) as a function of the one year default probability p_1 for a 45% loss given default λ (left) and as a function of the loss given default λ for a 1% default probability p_1 (right). The higher the default probability p_1 of the underlying credits and the higher the loss given default λ, the higher is the required regulatory capital k.

[29] For retail exposures the correlation coefficient ρ is set constant and equal to 15 percent.

[30] BCBS (2004, p. 68) describes the approaches for deriving an estimate of the portfolio effective maturity.

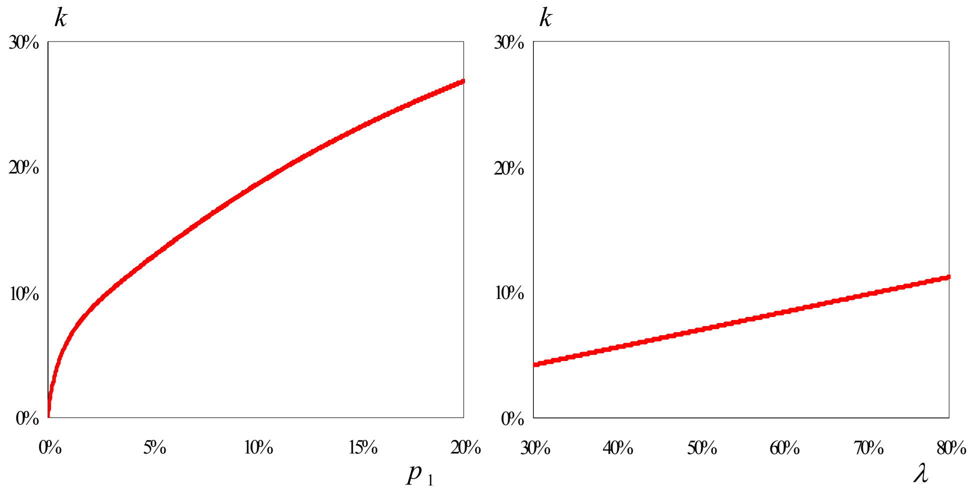

Figure 3: Capital requirements k as a function of the one year default probability p_1 for a 45% loss given default λ (left) and as a function of the loss given default λ for a 1% default probability p_1 (right).

For all the details regarding regulatory capital calculation see the original document BCBS (2004).[31] Finger (2001) and Gordy (2003) review the single factor model behind regulatory capital and discuss its calibration. Within the theoretical literature, Repullo and Suarez (2004) analyze the loan pricing implications of Basel II, and Elizalde and Repullo (2005) analyze the determinants of regulatory capital, economic and actual in the context of the single risk factor model outlined above.

[31] Additionally, the BCBS maintains a website with selected literature on concentration risk in credit portfolios: `www.bis.org/bcbs/events/rtf05biblio.htm`.

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